General Analysis of a Parallel-Plate Waveguide Inhomogeneously Filled with Gyromagnetic Media

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Abstract —The boundary value problem for a parallel-plate waveguide filled with inhomogeneous gyromagnetic medium is expressed and thoroughly examined in terms of a linear operator equation. A suitable vector definition of transverse mode functions is given and their completeness and orthogonality are proved. Applying a new set of continuity conditions for field components normal to the cross-sectional interface, the transfer matrix for the multilayered parallel-plate waveguiding structure is determined and used to formulate a characteristic equation. The analysis is illustrated by the numerical investigation of eigenfunctions and eigenvalues of a two layer ferrite–air guide.

I. Introduction

WAVEGUIDING structures inhomogeneously filled with isotropic or anisotropic—mainly gyrotropic media are of increasing interest for microwave and millimeter wave integration applications [1]–[10]. It has been found that simple methods of analysis of such structures do not lead to reliable results and therefore, the rigorous techniques of variational character should be applied in order to get an accurate solution.

In general, uniform guides like slot-microstrip lines or image guides, etc. and components constructed from these waveguides consist of a number of homogeneous layers with various types of planes (electric or magnetic) located at the interfaces between the layers or in the field symmetry planes. In most cases, subdivision of cross-sectional geometry of such structures leads to constituent subregions which can be considered as multilayered parallel-plate lines. This recognition is the fundamental concept of various exact methods in which, each such part of the waveguide is analyzed separately and the solution for the complete line is obtained by consequent matching of tangential fields at the common boundaries.

The above-mentioned approach requires the ability to represent the field components in each subregion in terms of a complete set of transverse mode functions (eigenfunctions). As far as dielectric structures are concerned this set of transverse mode functions is easy to determine by solving an appropriate Sturm–Liouville eigenvalue problem [14]. Since the theory of this eigenvalue problem is well developed one can formulate general properties of eigenvalues and eigenfunctions of arbitrarily filled isotropic waveguide. In particular, it is possible to show that in such structures the electromagnetic field can be expressed in terms of independent, complete sets of transverse mode functions, namely E and H, and that the mode functions of the same type are mutually orthogonal.

However, these conclusions are no longer valid for the structures containing gyrotropic media. Therefore up until now, very few [3]–[7] papers have been published concerning the investigation of similar guides unless they are homogeneously filled in the magnetization direction [8]–[10]. To overcome theoretical problems the authors of the papers [5]–[7] intuitively assume that it is possible to express the fields in subregions of such structures in terms of transverse mode functions. However, as far as we know neither proper definition nor the most essential features, (i.e., completeness, orthogonality and linear independent of terms) of the set of eigenfunctions for structures filled with gyromagnetic media have been given yet. Additionally, such an intuitive approach leads to very intricate eigenvalue equations which are valid only for specific waveguide geometries.

The purpose of the investigation reported here is to define and examine the general properties of transverse mode functions of a wide class of parallel-plate waveguides filled with inhomogeneous gyromagnetic medium and magnetized perpendicularly to the bounding planes. The unifying concept which accomplishes this objective is the recognition that the boundary value problem for such structures can be alternatively formulated in terms of an operator equation. The domain of the operator is defined as a Hilbert space and the results of spectral theory of linear operators are applied. In consequence, a definition of transverse mode functions is introduced and their properties are examined.

In particular it is found that:

1) the eigenfunctions of the waveguides inhomogeneously filled with gyromagnetic media should be defined as a vector with two constituents $(D, H)_x$ (x-direction of magnetization);
2) the set of eigenfunctions is complete;
3) due to self-adjointness of the eigenvalue problem all eigenvalues are real;

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4) all eigenfunctions are mutually orthogonal if an appropriate definition of inner product is applied.

Additionally, the new set of continuity conditions for normal field components is formulated and used to derive the eigenvalue equation for an arbitrary multilayered structure containing gyromagnetic material. The developed analysis is illustrated by the numerical investigation of eigenvalues and eigenfunctions of a ferrite–air parallel-plate waveguide.

The obtained results give a rigorous mathematical foundation for further analysis of complex ferrite-filled structures.

II. Formulation of the Problem

Let us consider a parallel-plate waveguide shown in the Fig. 1. The structure is filled with the gyromagnetic medium magnetized along the x-direction with static internal magnetic field H_t, bounded by either electric or magnetic walls placed in the planes $x = x_0$ and $x = x_N$ and laterally open. It is assumed that the gyromagnetic medium described by scalar relative permittivity $\epsilon(x)$ and tensor relative permeability $\vec{\mu}(x)$ is lossless but it may be inhomogeneous with respect to x-direction. Thus, both ϵ and $\vec{\mu}$ are the functions of position. Time harmonic variation of the form $e^{j\omega t}$ will be discussed in this study.

The electro-magnetic wave propagation in such a structure is governed by the following boundary value problem

$$\operatorname{curl}\frac{\vec{D}}{\epsilon(x)} = -jk_0\vec{\mu}(x)\vec{H}; \quad \operatorname{curl}\vec{H} = jk_0\vec{D} \qquad (1\text{a,b})$$

$$\operatorname{div}\vec{D} = 0; \qquad \operatorname{div}\left[\vec{\mu}(x)\vec{H}\right] = 0 \qquad (1\text{c,d})$$

$$B(\vec{D},\vec{H}) = 0|_{x=x_0,x_N} \qquad (1\text{e})$$

where $\vec{D}$ is the electric flux density, $\vec{H}$ is the normalized magnetic field ($\vec{H} = \eta_0\vec{H}$, η_0—the intrinsic impedance of the free space), and k_0 is the wavenumber in the free space. The relative permeability tensor $\vec{\mu}(x)$ has the form

$$\vec{\mu}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu_1(x) & -j\mu_2(x) \\ 0 & j\mu_2(x) & \mu_1(x) \end{bmatrix}.$$

In (1) $B(\vec{D},\vec{H}) = 0$ is the required boundary condition on $x = x_0$ and $x = x_N$. In case of a magnetic wall it can be written as follows:

$$B(\vec{D},\vec{H}) = B^H(\vec{D},\vec{H}) = \vec{a}_x\cdot\begin{bmatrix} 1 & 0 \\ 0 & \nabla_x \end{bmatrix}\begin{bmatrix} \vec{D} \\ \vec{H} \end{bmatrix} = 0. \qquad (2\text{a})$$

Similarly, in case of an electric wall

$$B(\vec{D},\vec{H}) = B^E(\vec{D},\vec{H}) = \vec{a}_x\cdot\begin{bmatrix} \nabla_x & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \vec{D} \\ \vec{H} \end{bmatrix} = 0 \qquad (2\text{b})$$

where ∇_x is a scalar operator, representing the partial derivative $\partial/\partial x$.

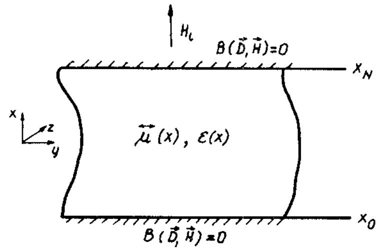
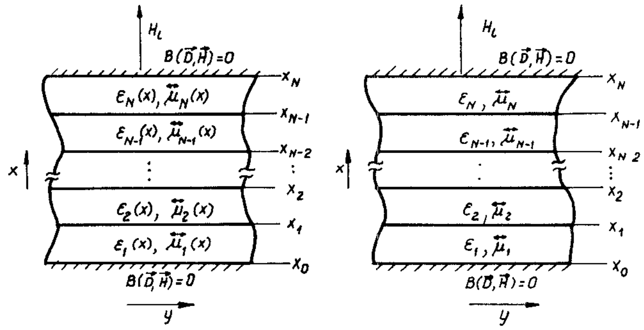


Fig. 1. A general structure of a parallel-plate line filled with inhomogeneous gyromagnetic medium.



(a) (b)

Fig. 2. Cross section of the multilayered gyromagnetic medium filled parallel-plate line. (a) $\epsilon(x)$ and $\vec{\mu}(x)$ are continuous functions inside each layer. (b) $\epsilon(x)$ and $\vec{\mu}(x)$ are constant inside each layer.

For generality of analytical formulation, we additionally assume that all functions involved in (1) are piecewise continuous on the interval (x_0, x_N). This assumption enables us to analyze a very wide class of waveguiding structures, namely:

a) layered waveguides with continuous variation of medium parameters inside each layer (Fig. 2(a));
b) layered waveguides whose medium parameters are constant inside each layer (Fig. 2(b));
c) waveguides whose parameters are continuous functions of x.

A. Vector Wave Equation

In order to transform the bounday value problem (1) into an alternative operator form which is more convenient for analytical treatment, the vector wave equation will be derived from Maxwell's equations.

At this stage of analysis we make temporarily an additional assumption that all functions of x have continuous derivatives on the interval (x_0, x_N) which means that only one class of waveguides, namely c, is considered. This restriction cancels, for the time being, the assumption made in the previous section, i.e., that all functions in (1) are piecewise continuous. However, such a restriction is very convenient from the analytical point of view because it ensures that all the combination of functions can be differentiated and therefore enables the necessary differential transforms to be unconditionally performed. This constraint will be discussed in detail in Section III-B.

As the structure under investigation is infinite and laterally open we may put

$$\vec{D} = \vec{D}(x, y, z) = \vec{D}(x) \cdot e^{k_y y + k_z z}$$

and

$$\vec{\tilde{H}} = \vec{\tilde{H}}(x, y, z) = \vec{\tilde{H}}(x) \cdot e^{k_y y + k_z z} \tag{3}$$

where k_y and k_z are the wavenumbers in y- and z-directions, respectively. Substituting (3) into Maxwell's equations and carrying-through some algebraic and differential manipulations (see Appendix A), we get the following vector wave equation:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} D_x(x) \\ \tilde{H}_x(x) \end{bmatrix} = 0 \tag{4}$$

where

$$A_{11} = \epsilon(x) \nabla_x \left[\frac{1}{\epsilon(x)} \nabla_x \right] + k_0^2 \mu_{\text{eff}}(x) \epsilon(x) + k_y^2 + k_z^2$$

$$A_{12} = -k_0 \epsilon(x) \frac{\mu_2(x)}{\mu_1(x)} \nabla_x$$

$$A_{21} = \nabla_x \left[k_0 \frac{\mu_2(x)}{\mu_1(x)} \right]$$

$$A_{22} = \nabla_x \left[\frac{1}{\mu_1(x)} \nabla_x \right] + k_y^2 + k_z^2 + k_0^2 \epsilon(x)$$

with $\mu_{\text{eff}} = [\mu_1^2(x) - \mu_2^2(x)]/\mu_1(x)$. For the sake of clarity, the y and z dependence $\exp(k_z z + k_y y)$ was suppressed in the above set of equations and henceforth will be omitted in all expressions.

The above equation was derived under the assumption that permeability and permittivity are continuous functions of x. However, as will be shown later, this equation is still valid if $\epsilon(x)$ and $\vec{\mu}(x)$ are piecewise continuous. Therefore, formula (4) defines the vector wave equation for arbitrary parallel-plate structure filled with inhomogeneous gyromagnetic medium described by $\epsilon(x)$ and $\vec{\mu}(x)$ magnetized along x-direction. From the equation it is seen that E_x and H_x waves are coupled. This means that even the unbounded gyromagnetic media may carry only coupled hybrid EH_x or HE_x modes. In the isotropic limit, when $\mu_1 = 1$, $\mu_2 = 0$, the first equation in (4) becomes Helmholtz's equation for E_x modes whereas the second is reduced to that for H_x modes given by Collin [14].

III. Operator Formulation

The boundary value problem (1) will be transformed in this section into an alternative operator form. In order to solve the resulting operator equation, the methods of functional analysis will be applied. For that purpose it is necessary to define the domain of the operator in terms of Hilbert space. It means that this domain has to be determined as a complete function space with an appropriate definition of inner product [11], [12].

At this stage, we still assume that all functions in (1) are continuous on the interval (x_0, x_N). Using the vector wave equation derived in the previous section the boundary value problem formulated by equations (1a)–(1e) takes the following form:

$$\boldsymbol{L}\underline{\varphi} + \delta\underline{\varphi} = 0 \tag{5a}$$

$$B(\underline{\varphi})|_{x = x_0, x_N} = 0 \tag{5b}$$

where

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{L}_{11} & \boldsymbol{L}_{12} \\ \boldsymbol{L}_{21} & \boldsymbol{L}_{22} \end{bmatrix} \tag{6}$$

$$\underline{\varphi} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} D_x(x) \\ \tilde{H}_x(x) \end{bmatrix}$$

and

$$\delta = k_y^2 + k_z^2 \tag{7}$$

with

$$\boldsymbol{L}_{11} = \epsilon(x) \nabla_x \left[\frac{1}{\epsilon(x)} \nabla_x \right] + k_0^2 \mu_{\text{eff}}(x) \epsilon(x)$$

$$\boldsymbol{L}_{12} = -k_0 \epsilon(x) \frac{\mu_2(x)}{\mu_1(x)} \nabla_x$$

$$\boldsymbol{L}_{21} = \nabla_x \left[k_0 \frac{\mu_2(x)}{\mu_1(x)} \right]$$

$$\boldsymbol{L}_{22} = \nabla_x \left[\frac{1}{\mu_1(x)} \nabla_x \right] + k_0^2 \epsilon(x).$$

In (5) both component functions φ_1 and φ_2 have physical meaning, which is to say their energy is finite. In other terms they belong to the complete $L^2(x_0, x_N)$ space. Simultaneously, according to the assumption formulated in the previous section, these functions and their derivatives are continuous for $x_0 \leqslant x \leqslant x_N$. Let Ω_1 denote the intersection of the above mentioned spaces then the domain of $\boldsymbol{L}$ can be defined as a cartesian product of Ω_1. Formally, it can be written as follows:

$$\underline{\varphi} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \in \Omega = \Omega_1 \times \Omega_1.$$

In other terms Ω is the vector space whose elements are differentiable functions of finite energy with continuous derivatives.

Having defined the domain of $\boldsymbol{L}$ we can introduce for this space the definition of inner product and then examine the properties of $\boldsymbol{L}$ in constructed in that way Hilbert space. For two arbitrary functions $\underline{u} = (u_1, u_2)^T$ and $\underline{v} = (v_1, v_2)^T$ from Ω we introduce the definition of inner product in the form

$$\langle \underline{u}, \underline{v} \rangle \overset{\text{df}}{=} \int_{x_0}^{x_N} \left[\frac{1}{\epsilon(x)} u_1 v_1 + u_2 v_2 \right] dx. \tag{8}$$

It is evident that the operator $\boldsymbol{L}$ which is determined by (6) is linear. Additionally, it can be shown (Appendix B) that for any combination of boundary conditions (2a) and (2b) operator $\boldsymbol{L}$ is symmetric. This leads to the conclusion that $\boldsymbol{L}$ is a self-adjoint operator [11], [12].

At the next step of analysis we will find the solution of alternative boundary value problem (5). Having formulated it in terms of operator equation, we can apply now the results of the spectral theory of linear operators [11], [12]. In particular, for equation (5) the following is true:

1) the operator equation (5) has solution for infinite number of discrete eigenvalues δ_l;
2) due to self-adjointness of $\boldsymbol{L}$ all eigenvalues are real (therefore, all eigenfunctions $\underline{\varphi}^l$ may be chosen to be real);
3) the set of eigenfunctions $\{\underline{\varphi}^l\}$ is dense in Ω, i.e., is the complete set of functions in that space;
4) any two distinct eigenfunctions of (5) are orthogonal; it means that
$$(\delta_l - \delta_k)\langle \underline{\varphi}^l, \underline{\varphi}^k \rangle = 0$$
for each values l and k;
5) the eigenvalues can be ordered such that
$$\delta_0 < \delta_1 < \cdots < \delta_l < \delta_{l+1} < \cdots.$$

A. Transverse Mode Functions

According to the definition of the domain of the operator $\boldsymbol{L}$, every eigenfunction $\underline{\varphi}^l$ is a vector with two constituent elements $\varphi_1^{(l)}$ and $\varphi_2^{(l)}$ which correspond to $D_x^{(l)}(x)$ and $\tilde{H}_x^{(l)}(x)$, respectively. Therefore, the eigenfunctions $\{\underline{\varphi}^l\}$ represent simultaneously the transverse mode functions of a parallel-plate waveguide filled with inhomogeneous gyromagnetic medium magnetized along x-direction.

From 3) one may conclude that any transverse field can be expressed in terms of modal solutions. Moreover, these transverse mode functions are mutually orthogonal. Bearing in mind (8) and 4), the orthogonality relation for transverse mode functions can be written as follows:

$$\int_{x_0}^{x_N} \left[\frac{1}{\epsilon(x)} D_x^l(x) D_x^k(x) + \tilde{H}_x^l(x) \tilde{H}_x^k(x) \right] dx = N_{lk} \delta_{lk}. \tag{9}$$

Here, N_{lk} is a normalization coefficient and δ_{lk} stands for Kronecker delta.

If the wave equation (4) can be separated into a pair of scalar Helmholtz's equations (isotropic case) then each component of the above integral vanishes independently and one gets two well known orthogonality relations for E_x and H_x modes [1].

The obtained result seems very important and useful. In particular, it guarantees that transverse mode functions of hybrid modes are linearly independent and therefore can serve in various variational methods as a set of basis functions.

B. Continuity Conditions for Transverse Mode Functions

In this section we will extend the above developed analysis to the wider class of problems when $\epsilon(x)$ and $\vec{\mu}(x)$ are piecewise continuous.

Let us assume that the interval (x_0, x_N) has been divided into N-parts and $\epsilon(x)$ and $\vec{\mu}(x)$ are continuous on the interval (x_{i-1}, x_i) $i = 1 \cdots N$ (Fig. 2a). In this case $\epsilon(x)$ and $\vec{\mu}(x)$ can be expressed in the following manner:

$$\begin{Bmatrix} \epsilon(x) \\ \vec{\mu}(x) \end{Bmatrix} = \sum_{i=1}^{N} \begin{Bmatrix} \epsilon_i(x) \\ \vec{\mu}_i(x) \end{Bmatrix} \cdot \mathscr{X}_i.$$

Here, $\mathscr{X}_i$ is the characteristic function of ith region

$$\mathscr{X}_i = 1, \quad \text{for } x_{i-1} \leqslant x < x_i$$
$$= 0, \quad \text{elsewhere.}$$

To answer the question whether the hitherto developed analysis still holds we must come back to Section II where the wave equation was derived. In order to enable us to perform the necessary differential operations, we assumed there that all functions and their derivatives are continuous on the interval (x_0, x_N). As a matter of fact this restriction is too severe because it suffices to require the expressions to which operator ∇_x is applied, to be continuous.

Hence, considering equations (A.10) and (A.12) given in Appendix A we can formulate the following necessary continuity conditions:

$$D_x^{(i)}(x)|_{x=x_i^-} = D_x^{(i+1)}(x)|_{x=x_i^+} \tag{10a}$$

$$\frac{1}{\epsilon_i(x)} \nabla_x D_x^{(i)}(x)|_{x=x_i^-} = \frac{1}{\epsilon_{i+1}(x)} \nabla_x D_x^{(i+1)}(x)|_{x=x_i^+} \tag{10b}$$

$$\tilde{H}_x^{(i)}(x)|_{x=x_i^-} = \tilde{H}_x^{(i+1)}(x)|_{x=x_i^+} \tag{10c}$$

$$\left[\frac{\mu_{2_i}(x)}{\mu_{1_i}(x)} k_0 D_x^{(i)}(x) + \frac{1}{\mu_{1_i}(x)} \nabla_x \tilde{H}_x^{(i)}(x) \right] \Bigg|_{x=x_i^-} = \left[\frac{\mu_{2_{i+1}}(x)}{\mu_{1_{i+1}}(x)} k_0 D_x^{(i+1)}(x) + \frac{1}{\mu_{1_{i+1}}(x)} \nabla_x \tilde{H}_x^{(i+1)}(x) \right] \Bigg|_{x=x_i^+} \tag{10d}$$

Note, that although the interface between different magnetic media is considered, the normal $\tilde{H}_x$ component is continuous across the boundary. This results from the form of the relative permeability tensor $\vec{\mu}$ defined in Section II in which, first diagonal element is equal one. If the restrictions (10) are fulfilled, then the boundary value problem (1) can be alternatively expressed in terms of the operator equation (5) with linear operator $\boldsymbol{L}$ defined by (6). Simultaneously, it is possible to show that the proof of the symmetry of the operator $\boldsymbol{L}$ given in Appendix B is still valid.

Let us examine now whether these necessary conditions are at the same time the sufficient conditions. In other words we will investigate if they lead to the continuity of tangential components at the cross-sectional interface.

We may assume that the wavenumbers $k_y^{(i)}$ and $k_z^{(i)}$ are the same in each layer. For ith cross-section interface we put $k_y^{(i)} = k_y^{(i+1)} = k_y$ and $k_z^{(i)} = k_z^{(i+1)} = k_z$. Using (A.3), (A.6), (A.8) together with (A.11) and inserting into (10), we have

$$\frac{D_t^{(i)}(x)}{\epsilon_i(x)} \Bigg|_{x=x_i^-} = \frac{D_t^{(i+1)}(x)}{\epsilon_{i+1}(x)} \Bigg|_{x=x_i^+}$$

$$\tilde{H}_t^{(i)}(x)|_{x=x_i^-} = \tilde{H}_t^{(i+1)}(x)|_{x=x_i^+}. \tag{11}$$

Here, t denotes components tangential to the cross-sectional interface. This implies that the conditions (10) are entirely equivalent to (11). Thus, we have proved that the equations (10) form the set of necessary and at the same time sufficient conditions for normal field components D_x and $\tilde{H}_x$. If they are fulfilled then the tangential fields are simultaneously matched.

IV. Eigenfunctions and Eigenvalues of a Parallel-Plate Waveguide Filled with Layers of Homogeneous Gyromagnetic Medium

Presently, in order to illustrate the developed analysis we will show in detail how to determine transverse mode functions when ϵ and $\vec{\mu}$ are homogeneous inside each interval, i.e., $\epsilon_i(x)=\epsilon_i$, $\vec{\mu}_i(x)=\vec{\mu}_i$. In this case one gets the structure of parallel-plate line presented in the Fig. 2(b).

First we will find the general solutions of Maxwell's equations in ith layer. From (4) we get the wave equation

$$\begin{bmatrix} \nabla_x^2 + k_0^2\mu_{\text{eff}_i}\epsilon_i + \delta & -k_0\epsilon_i\dfrac{\mu_{2_i}}{\mu_{1_i}}\nabla_x \\ k_0\dfrac{\mu_{2_i}}{\mu_{1_i}}\nabla_x & \dfrac{1}{\mu_{1_i}}\nabla_x^2 + k_0^2\epsilon_i + \delta \end{bmatrix} \begin{bmatrix} D_x^{(i)}(x) \\ \tilde{H}_x^{(i)}(x) \end{bmatrix} = 0 \tag{12}$$

here $\delta = k_y^2 + k_z^2$.

The general solution of the above linear, differential equation can be expressed as follows:

$$D_x^{(i)}(x) = \sum_{k=1}^{4} A_k^{(i)} S_k^{(i)} \exp \lambda_k^{(i)}(x - x_{i-1})$$

$$\tilde{H}_x^{(i)}(x) = \sum_{k=1}^{4} A_k^{(i)} R_k^{(i)} \exp \lambda_k^{(i)}(x - x_{i-1}) \tag{13}$$

where $A_k^{(i)}$ $k=1$ to 4 are unknown constants whereas $S_k^{(i)}$ and $R_k^{(i)}$ are elements of fundamental matrix for (12) and $\lambda_k^{(i)}$ are roots of its characteristic equation.

Substituting (13) into (12) we get

$$\lambda_1^{(i)} = -\lambda_2^{(i)} = \left\{\tfrac{1}{2}\left[g_2 + \left(g_2^2 - 4g_0\right)^{1/2}\right]\right\}^{1/2}$$

$$\lambda_3^{(i)} = -\lambda_4^{(i)} = \left\{\tfrac{1}{2}\left[g_2 - \left(g_2^2 - 4g_0\right)^{1/2}\right]\right\}^{1/2} \tag{14}$$

where

$$g_2 = -\delta(1+\mu_{1_i}) - 2k_0^2\epsilon_i\mu_{1_i}$$

$$g_0 = \mu_{1_i}(\delta + k_0^2\epsilon_i)(\delta + k_0^2\epsilon_i\mu_{\text{eff}_i}). \tag{15}$$

All particular solutions of (12) are linearly independent, therefore we may put

$$R_1^{(i)} = R_2^{(i)} = S_3^{(i)} = S_4^{(i)} = 1. \tag{16a}$$

This implies

$$R_3^{(i)} = -R_4^{(i)} = \frac{\mu_{1_i}}{k_0\epsilon_i\mu_{2_i}\lambda_3^{(i)}}\left(\lambda_3^{(i)^2} + \delta + k_0^2\epsilon_i\mu_{\text{eff}_i}\right)$$

$$S_1^{(i)} = -S_2^{(i)} = -\frac{1}{k_0\mu_{2_i}\lambda_1^{(i)}}\left(\lambda_1^{(i)^2} + \mu_{1_i}(\delta + k_0^2\epsilon_i)\right). \tag{16b}$$

Such a choice of R_ks and S_ks guarantees that R_3 and S_1, which may be interpreted as coupling coefficients, vanish for isotropic media, and in this case we get two uncoupled waves.

Having defined the general solution of Maxwell's equations in each layer we may determine the characteristic equation for eigenvalues of the considered boundary value problem. For this purpose we will apply the transverse resonance method [9], [13], but instead of matching tangential fields we will use the continuity conditions (10). To define the transfer matrix it is convenient to introduce a continuity vector $\underline{F}$. According to (10), we may put

$$\underline{F}^{(i)} = \left[F_1^{(i)}, F_2^{(i)}, F_3^{(i)}, F_4^{(i)}\right]^T = \begin{bmatrix} \dfrac{1}{\epsilon_i}\nabla_x & 0 & 1 & \dfrac{\mu_{2_i}}{\mu_{1_i}}k_0 \\ 0 & 1 & 0 & \dfrac{1}{\mu_{1_i}}\nabla_x \end{bmatrix}^T \begin{bmatrix} D_x^{(i)} \\ \tilde{H}_x^{(i)} \end{bmatrix}. \tag{17}$$

Subsequently, using the above equation together with (13) we define the transfer matrix $\underline{\underline{T}}^{(i)}$ which links the continuity vector $\underline{F}^{(i)}$ between the extreme heights x_i^-, x_{i-1}^+ of the ith layer.

$$\underline{F}^{(i)}|_{x=x_i^-} = \underline{\underline{T}}^{(i)}\underline{F}^{(i)}|_{x=x_{i-1}^+} \tag{18}$$

where

$$\underline{\underline{T}}^{(i)} = \underline{\underline{Q}}_i\left[\text{diag}\left(\exp\lambda_k^{(i)}(x_i - x_{i-1})\right)\right]\underline{\underline{Q}}_i^{-1}, \qquad k = 1\cdots 4. \tag{19}$$

The elements of the matrix $\underline{\underline{Q}}_i$ take up the form

$$Q_{1k}^{(i)} = \frac{1}{\epsilon_i}\lambda_k^{(i)}S_k^{(i)}$$

$$Q_{2k}^{(i)} = R_k^{(i)}$$

$$Q_{3k}^{(i)} = S_k^{(i)}$$

$$Q_{4k}^{(i)} = \frac{\mu_{2_i}}{\mu_{1_i}}k_0 S_k^{(i)} + \frac{1}{\mu_{1_i}}\lambda_k^{(i)}R_k^{(i)}. \tag{20}$$

Using continuity conditions (10) and the transfer matrix notation (19) one can express the transfer of the continuity vector $\underline{F}$ through the multilayered structure in terms of global transfer matrix $\underline{\underline{T}}$

$$F^{(N)}|_{x=x_N} = \prod_{i=N}^{1} \underline{\underline{T}}^{(i)}\underline{F}^{(1)}|_{x=x_0} = \underline{\underline{T}}_{4\times 4}\underline{F}^{(1)}|_{x=x_0}. \tag{21}$$

Additionally, in order to fulfill the boundary conditions at the upper $x=x_N$ and lower $x=x_0$ screen, it suffices to require

$$\left.\begin{array}{ll} F_1^{(i)} = F_2^{(i)} = 0, & \text{for an electric wall} \\ F_3^{(i)} = F_4^{(i)} = 0, & \text{for a magnetic wall} \end{array}\right\} \quad i = 1 \text{ or } N.$$

Inserting (22) into (21) we get the set of linear algebraic equations. According to the combination of boundary con-

TABLE I
THE FORM OF THE CHARACTERISTIC EQUATION FOR DIFFERENT COMBINATION OF BOUNDARY CONDITIONS ($\underline{T}_{ik}$ are the SUBMATRICES OF THE GLOBAL TRANSFER MATRIX $\underline{\underline{T}}$)

boundary conditions at		characteristic equation
$x = x_0$	$x = x_N$	
$B^E(\vec{D}, \vec{H}) = 0$	$B^E(\vec{D}, \vec{H}) = 0$	$\det \underline{\underline{T}}_{12} = 0$
$B^E(\vec{D}, \vec{H}) = 0$	$B^H(\vec{D}, \vec{H}) = 0$	$\det \underline{\underline{T}}_{11} = 0$
$B^H(\vec{D}, \vec{H}) = 0$	$B^E(\vec{D}, \vec{H}) = 0$	$\det \underline{\underline{T}}_{22} = 0$
$B^H(\vec{D}, \vec{H}) = 0$	$B^H(\vec{D}, \vec{H}) = 0$	$\det \underline{\underline{T}}_{21} = 0$

ditions at $x = x_0$ and $x = x_N$, the characteristic or eigenvalue equation takes up one of the forms given in Table I.

The solution of the set of linear equations obtained from (21) under the condition of vanishing of the determinant of an appropriate submatrix $\underline{T}_{ik}$ allows to determine (via (18) and (10)) the field amplitudes $A_k^{(i)}$ ($k = 1$ to 4, $i = 1$ to N). Once these coefficients are evaluated, the transverse mode function for a layered waveguide is constructed.

V. NUMERICAL EXAMPLE

In this section the results of the numerical investigation of eigenvalues and eigenfunctions of a ferrite–air parallel-plate waveguide are presented. In this case the characteristic equation takes up the following form:

$$S_1 R_3 \left(\lambda_0 \bar{C}_1 \bar{S}_0 + \frac{\lambda_1}{\epsilon_f} \bar{S}_1 \bar{C}_0 \right) \left(\frac{\lambda_3}{\mu_1} \bar{C}_3 \bar{S}_0 + \lambda_0 \bar{S}_3 \bar{C}_0 \right) + \frac{\bar{S}_0 \bar{C}_0}{\epsilon_f \mu_1} \mu_2 k_0 S_1 \left(\lambda_1 \bar{S}_1 \bar{C}_3 - \lambda_3 \bar{C}_1 \bar{S}_3 \right) - \left(\lambda_0 \bar{S}_1 \bar{C}_0 + \frac{\lambda_1}{\mu_1} \bar{C}_1 \bar{S}_0 \right) \left(\frac{\lambda_3}{\epsilon_f} \bar{S}_3 \bar{C}_0 + \lambda_0 \bar{C}_3 \bar{S}_0 \right) = 0$$

with

$$\bar{S}_{1(3)} = \sinh(\lambda_{1(3)} d_f) \qquad \bar{S}_0 = \sinh(\lambda_0 d_a)$$
$$\bar{C}_{1(3)} = \cosh(\lambda_{1(3)} d_f) \qquad \bar{C}_0 = \cosh(\lambda_0 d_a)$$

where d_f, d_a is the height of ferrite and air layer, respectively; ϵ_f is the relative permittivity of a ferrite medium and $\lambda_0^2 = -k_0^2 - \delta$. From the above formula it is seen that for the isotropic limit (i.e., H_i tends to infinity; $\mu_1 \to 1$; $(S_1, R_3, \mu_2) \to 0$) one gets two independent characteristic equations for E_x and H_x modes similar to those given by Emert [2].

The frequency behavior of the eigenvalues of the linear operator (6) is shown in the Fig. 3. Numerical calculations have been carried out for the frequency range where $0 < \mu_1 < 1$. This region is attractive from the technical point of view because of the existence of the strong field displacement effect, in particular, in the frequency band in which $\mu_{\text{eff}} < 0$. Note, that under the condition $k_y = 0$, according

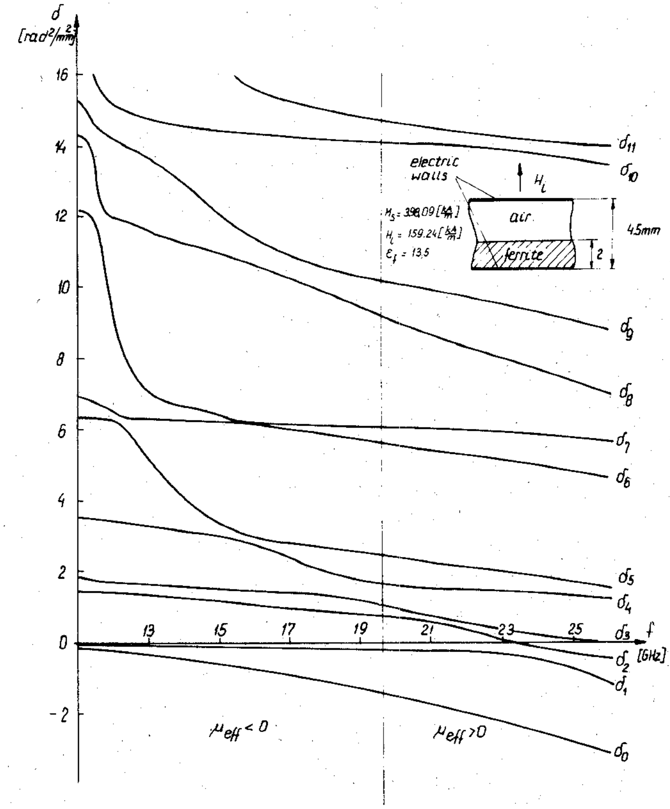


Fig. 3. Eigenvalues of the ferrite–air parallel-plate waveguide.

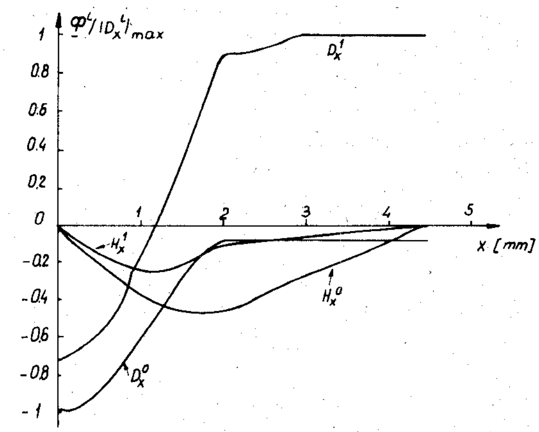


Fig. 4. Normalized field strength for two first eigenfunctions $\underline{\varphi}^0$ and $\underline{\varphi}^1$ of the line depicted in Fig. 3, ($f = 11$ GHz).

to the definition (7), the set of eigenvalues $\{\delta_i\}$ defines the modal spectrum $\{k_{z_i}^2\}$ of a parallel-plate line. From the diagram it is seen that the obtained results agree well with theoretical predictions. In particular, the curves $\delta(f)$ have no common points which means that the conclusion 5) given in Section II is valid. However, the curves approach one another which leads to the conclusion that there exists a cross-coupling among the modes of a ferrite–air parallel-plate line.

Additionally, two first eigenfunctions $\underline{\varphi}^0$ and $\underline{\varphi}^1$ are plotted in the Fig. 4. It is seen that for $\underline{\varphi}^0$ the $\tilde{H}_x$ component is dominant, therefore this eigenfunction corresponds to the $(HE^0)_x$ mode of the parallel-plate ferrite–air wave-

TABLE II
THE VALUES OF THE INNER PRODUCT FOR TWO FIRST EIGENFUNCTIONS

i	k	$\langle \underline{\varphi}^i, \underline{\varphi}^k \rangle / \langle \underline{\varphi}^1, \underline{\varphi}^1 \rangle$
0	0	$1.7 \cdot 10^{-1}$
1	0	$1.864 \cdot 10^{-6}$
1	1	1

guide. In the second case E_x is dominant, therefore $\underline{\varphi}^1$ corresponds to the $(EH^1)_x$ mode of the depicted line. Finally, in order to verify the orthogonality relation (9), the inner product for all the combinations of two first eigenfunctions was computed. The results are given in the Table II.

The finite value of $\langle \underline{\varphi}^1, \underline{\varphi}^0 \rangle$ can be easily explained by the accuracy of utilized numerical methods as well as by the precision of digital computations (seven significant digits).

CONCLUSIONS

General properties of transverse mode functions of a parallel-plate waveguide inhomogeneously filled with gyromagnetic media have been investigated. These functions which are the eigenfunctions of a linear operator have been found to create a complete, orthogonal functional set. Additionally, it has been shown that due to self-adjointness of the operator all the eigenvalues are real and the eigenfunctions may be chosen to be real. For layered structures, the new set of continuity conditions for field components normal to an interface, has been formulated and used to define the eigenvalue equation. The theoretical predictions have been confirmed by the results of the numerical investigation of the eigenvalues and eigenfunctions of a ferrite–air parallel-plate waveguide.

APPENDIX A

Derivation of a Vector Wave Equation

To derive a vector wave equation for inhomogeneously filled parallel-plate waveguide, it is convenient to assume that all functions involved in Maxwell's equations have continuous derivatives. Substituting (3) into (1a)–(1d) we get Maxwell's equations in the following form:

$$k_z \frac{D_x}{\epsilon(x)} - \nabla_x \frac{D_z}{\epsilon(x)} + jk_0\mu_1(x)\tilde{H}_y + k_0\mu_2(x)\tilde{H}_z = 0 \quad \text{(A.1)}$$

$$\nabla_x \frac{D_y}{\epsilon(x)} - k_y \frac{D_x}{\epsilon(x)} - k_0\mu_2(x)\tilde{H}_y + jk_0\mu_1(x)\tilde{H}_z = 0 \quad \text{(A.2)}$$

$$k_y \frac{D_z}{\epsilon(x)} - k_z \frac{D_y}{\epsilon(x)} + jk_0\tilde{H}_x = 0 \quad \text{(A.3)}$$

$$k_z\tilde{H}_x - \nabla_x\tilde{H}_z - jk_0D_y = 0 \quad \text{(A.4)}$$

$$\nabla_x\tilde{H}_y - k_y\tilde{H}_x - jk_0D_z = 0 \quad \text{(A.5)}$$

$$k_y\tilde{H}_z - k_z\tilde{H}_y - jk_0D_x = 0 \quad \text{(A.6)}$$

$$\mu_1(x)\left(k_y\tilde{H}_y + k_z\tilde{H}_z\right) - j\mu_2(x)\left(k_y\tilde{H}_z - k_z\tilde{H}_y\right) + \nabla_x\tilde{H}_x = 0 \quad \text{(A.7)}$$

$$k_yD_y + k_zD_z + \nabla_xD_x = 0. \quad \text{(A.8)}$$

Equation (A.1) together with (A.8) yields

$$k_z^2\frac{D_x}{\epsilon(x)} + \nabla_x\left[\frac{k_yD_y + \nabla_xD_x}{\epsilon(x)}\right] + jk_0k_z\mu_1(x)\tilde{H}_y + k_0k_z\mu_2(x)\tilde{H}_z = 0. \quad \text{(A.9)}$$

Next, using (A.2) and (A.7) and inserting the results into (A.9) one gets

$$\nabla_x\left[\frac{1}{\epsilon(x)}\nabla_xD_x\right] + k_0^2\mu_{\text{eff}}(x)D_x + \left(k_y^2 + k_z^2\right)\frac{D_x}{\epsilon(x)} - \frac{\mu_2(x)}{\mu_1(x)}k_0\nabla_x\tilde{H}_x = 0 \quad \text{(A.10)}$$

with

$$\mu_{\text{eff}}(x) = \frac{\mu_1^2(x) - \mu_2^2(x)}{\mu_1(x)}.$$

Similarly from (A.6) and (A.7) we obtain

$$k_y\tilde{H}_y + k_z\tilde{H}_z + \frac{\mu_2(x)}{\mu_1(x)}k_0D_x + \frac{1}{\mu_1(x)}\nabla_x\tilde{H}_x = 0. \quad \text{(A.11)}$$

Combining this equation with (A.3)–(A.5) we have

$$\nabla_x\left[\frac{1}{\mu_1(x)}\nabla_x\tilde{H}_x + k_0\frac{\mu_2(x)}{\mu_1(x)}D_x\right] + \left[\left(k_y^2 + k_z^2\right) + k_0^2\epsilon(x)\right]\tilde{H}_x = 0. \quad \text{(A.12)}$$

Equations (A.10) and (A.12) can be rewritten in vector form yielding the wave equation defined by (4).

APPENDIX B

Proof of the Symmetry of $\boldsymbol{L}$

For any combination of boundary conditions (2a) or (2b) with the inner product given by (8) operator $\boldsymbol{L}$ defined by (6) is symmetric, i.e.,

$$\langle \boldsymbol{L}\underline{u}, \underline{v}\rangle - \langle \underline{u}, \boldsymbol{L}\underline{v}\rangle = 0$$

for any $\underline{u}, \underline{v} \in \Omega$

Proof: According to (8) one gets

$$\langle \boldsymbol{L}\underline{u}, \underline{v}\rangle - \langle \underline{u}, \boldsymbol{L}\underline{v}\rangle$$

$$= \int_{x_0}^{x_N}\left\{\left(v_1\nabla_x\left[\frac{1}{\epsilon(x)}\nabla_x u_1\right] - u_1\nabla_x\left[\frac{1}{\epsilon(x)}\nabla_x v_1\right]\right)\right. \tag{B.1a}$$

$$+\left(v_1 k_0^2\mu_{\text{eff}}(x)u_1 - u_1 k_0^2\mu_{\text{eff}}(x)v_1\right) \tag{B.1b}$$

$$+\left(-v_1 k_0\frac{\mu_2(x)}{\mu_1(x)}\nabla_x u_2 + u_1 k_0\frac{\mu_2(x)}{\mu_1(x)}\nabla_x v_2\right) \tag{B.1c}$$

$$+\left(v_2\nabla_x\left[k_0\frac{\mu_2(x)}{\mu_1(x)}u_1\right] - u_2\nabla_x\left[k_0\frac{\mu_2(x)}{\mu_1(x)}v_1\right]\right) \tag{B.1d}$$

$$+\left(v_2\nabla_x\left[\frac{1}{\mu_1(x)}\nabla_x u_2\right] - u_2\nabla_x\left[\frac{1}{\mu_1(x)}\nabla_x v_2\right]\right) \tag{B.1e}$$

$$\left.+\left(v_2 k_0^2\epsilon(x)u_2 - u_2 k_0^2\epsilon(x)v_2\right)\right\}dx. \tag{B.1f}$$

In the above expression components (B.1b) and (B.1f) vanish. The remaining terms require integration by parts. For instance integration of the first term in (B.1a) gives

$$\int_{x_0}^{x_N} v_1\nabla_x\left[\frac{1}{\epsilon(x)}\nabla_x u_1\right]dx$$

$$= v_1\frac{1}{\epsilon(x)}\nabla_x u_1\Big|_{x_0}^{x_N} - \int_{x_0}^{x_N}\left(\frac{1}{\epsilon(x)}\nabla_x v_1\nabla_x u_1\right)dx. \tag{B.2}$$

According to (2) the first term in (B.2) vanishes for any combination of boundary conditions for $x = x_0$ and $x = x_N$. Applying the same procedure for remaining terms one gets

$$\langle \boldsymbol{L}\underline{u}, \underline{v}\rangle - \langle \underline{u}, \boldsymbol{L}\underline{v}\rangle$$

$$= \int_{x_0}^{x_N}\left\{-\frac{1}{\epsilon(x)}\nabla_x v_1\nabla_x u_1 + \frac{1}{\epsilon(x)}\nabla_x u_1\nabla_x v_1\right.$$

$$+ u_2\nabla_x\left[v_1 k_0\frac{\mu_2(x)}{\mu_1(x)}\right] - v_2\nabla_x\left[u_1 k_0\frac{\mu_2(x)}{\mu_1(x)}\right]$$

$$+ v_2\nabla_x\left[k_0\frac{\mu_2(x)}{\mu_1(x)}u_1\right] - u_2\nabla_x\left[k_0\frac{\mu_2(x)}{\mu_1(x)}v_1\right]$$

$$\left.-\frac{1}{\mu_1(x)}\nabla_x u_2\nabla_x v_2 + \frac{1}{\mu_1(x)}\nabla_x v_2\nabla_x u_2\right\}dx = 0.$$

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